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Linked-Cluster Expansion for the Graph-Vertex Coloration Problem*

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The logarithm of the chromatic polynomial is expanded in terms of the number of weak embeddings of star subgraphs of successively higher numbers of vertices. This expansion is used to estimate the number of colorations per vertex for a number of two- and three-dimensional space lattices. Finally, the expansion is further simplified in terms of the number of strong embeddings of a subset of the star subgraphs of successively higher numbers of vertices.

I. INTRODUCTION

The aim of this paper is to simplify the expression for the number of ways of coloring the vertices of a graph under the restriction that no two vertices joined by an edge may be given the same color. We shall use combinatorial methods common in statistical mechanics where problems involving large graphs of simple structure frequently occur. In particular, we shall consider not the chromatic polynomial, but its logarithm. This procedure has the advantage that a "linked-cluster" or star graph expansion can be given which is much simpler (that is far less combinatorial data on the structure of the graph is required) than was formerly required [1, 2, 3]. For this expansion only the number of embeddings of star graphs is required and not those for singly connected, articulated, bridged, or separated graphs. We prove that, when the expansion includes all stars of m or fewer vertices, if we use p-colors then the error we make is of the order of p^{-m}. We have used this expansion to estimate the number of colorations per vertex for the square, honeycomb, diamond, simple cubic, and triangular lattices. In addition, a rough estimate is given for the number of ways of coloring a special class of planar graphs. In the final

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section we show how the use of the strong embedding system of graph constants restricts the class graphs we need to consider to only a certain subset of the star graphs.

II. Simplification of the Series

Nagle [1] has recently drawn attention to the fact that the problem of computing the number of distinct colorations of a graph with p colors can be considered through the series expansion techniques used in statistical mechanics. In particular, he rederived the result of Birkhoff [2] and Whitney [3] for the chromatic polynomial.

Starting in a manner quite parallel to Nagle [1], we proceed as follows. Let

$$W_p(G) = \operatorname{tr}\left\{ \prod_{\text{bonds of } G} [I - \delta_{\mu_i\mu_j}] \right\} \tag{2.1}$$

be the number of distinct ways of coloring a graph G with p colors. We denote the vertices of G by $i, j \cdots$. The color at vertex i is denoted by μ_i. The operator, I, is the identity operator and $\delta_{\mu_i\mu_j}$ is a diagonal matrix which has the value 1 if $\mu_i = \mu_j$, and zero otherwise. The trace is to be taken over the p^n dimensional space of p colors for each of the n vertices of G. Equation (2.1) then gives the number of distinct ways of coloring G when we prohibit the same color from appearing at both ends of a single edge. The usual chromatic polynomial results from an expansion of (2.1) and evaluation of the trace [1]:

$$W_p(G) = \operatorname{tr}\left\{ I - \sum_{\{ij\}} \delta_{\mu_i\mu_j} + \frac{1}{2!} \sum_{\{ij \neq kl\}} \delta_{\mu_i\mu_j}\delta_{\mu_k\mu_l} - \cdots \right\} \tag{2.2}$$

We shall now proceed to develop the linked-cluster expansion for $\ln W_p(G)$. First let us define

$$Z_p(G) = W_p(G)/\operatorname{tr}\{I\} \tag{2.3}$$

where the trace in the denominator is always taken over the same space as that used to define the numerator. In order to make contact with the exposition of Rushbrooke [4] we rewrite

$$I - \delta_{\mu_i\mu_j} = \lim_{x\to\infty} \exp[-x\delta_{\mu_i\mu_j}]. \tag{2.4}$$

Thus, if we define

$$\mathscr{H} = \sum_{i,j} \delta_{\mu_i\mu_j}, \tag{2.5}$$

then with the observation

$$Z_p(G) = \lim_{x\to\infty} \operatorname{tr}\{e^{-x\mathscr{H}}\}/\operatorname{tr}\{I\} \tag{2.6}$$

we have cast (before $x \to \infty$) the coloration problem into the general form which he [4] considers. For problems of this form he establishes the following theorem:

THEOREM 1 (Rushbrooke).

$$\ln Z_p(G, x) = \sum_\tau T_{G,\tau} f_\tau(x, p). \tag{2.7}$$

$T_{G,\tau}$ is the number of distinct ways an unlabeled subgraph τ can be (weakly) embedded on G such that every edge of τ corresponds to an edge of G. The summation is over the set of all star subgraphs, τ—i.e., those which are multiply connected plus the graph consisting of a single edge.

That the τ must be multiply connected instead of only connected, as Rushbrooke [4] shows for the general case, follows because, if a graph $\bar{\tau}$ has an articulation point, then if we divide it into two section graphs $\bar{\tau}_1$ and $\bar{\tau}_2$ by cutting it at that point, we can easily show that

$$\ln Z_p(\bar{\tau}, x) = \ln Z_p(\bar{\tau}_1, x) + \ln Z_p(\bar{\tau}_2, x).$$

This relation is the same one used by Rushbrooke [4] to show that the contribution f from graphs with separated components vanishes. Thus the contribution also vanishes from graphs with an articulation point. The functions $f_\tau(x, p)$ depend only the subgraphs τ and not at all on G.

This expression represents a very considerable simplification in the embedding data required over (2.2) as the number of configurations to be enumerated is very much less. Only the star graphs remain; the articulated, bridged, and separated graphs are eliminated. The limit as $x \to \infty$ is easily taken. Rushbrooke [4] further points out that matrix $T_{G,\tau}$ can be made triangular with $\det |T| = 1$, by arranging the τ in increasing order of size (a larger—more edges and or more vertices—graph fits zero times on a smaller one and once on itself). Thus we may write by (2.6) and (2.7)

$$f_\tau(p) = \lim_{x\to\infty} f_\tau(x, p) = \sum_{\tau'} (T^{-1})_{\tau,\tau'} [\ln Z_p(\tau')], \tag{2.8}$$

as T^{-1} is independent of x, and the $\sum_{\tau}$, is a finite sum. Consequently, as the $f_\tau(p)$ are well-defined functions of p, we have the result.

THEOREM 2.

$$\ln Z_p(G) = \sum T_{G,\tau} f_\tau(p). \tag{2.9}$$

Here the notation is as in Theorem 1 *and the sum is only over star subgraphs* τ.

We now establish the following useful property of the f's.

THEOREM 3.

$$f_\tau(p) = O(p^{-(n_\tau - 1)}). \tag{2.10}$$

Here n_τ *is the number of vertices in* τ *and the above relation holds for large positive real* p.

To obtain this result, consider again (2.2). As Nagle [1] shows, this sum may be organized as

$$W_p(G) = p^n \sum_{\mathbf{d},t} g(\mathbf{d}, t)(-1)^e (p)^{t-v}, \tag{2.11}$$

where e, v, t are the edges, vertices, and separated components. The function $g(\mathbf{d}, t)$ is the number of subgraphs of G with partial description $\mathbf{d}$ and t separated components. If we now truncate expansion (2.11) and take only those terms with $v - t \leqslant m$, then the corresponding truncated

$$\tilde{Z}_p(G) = p^{-n} \tilde{W}_p(G)$$

will be exact through order p^{-m}. As the methods of Rushbrooke [4] used to establish (2.7) and our subsequent (2.9) are formal identities, $\ln Z_p(G)$ formed from the truncated series will also be exact as a function of p^{-1} to order p^{-m}. However, the selection procedure used corresponds exactly to restricting τ in (2.9) to graphs of $\leqslant m + 1$ points. Hence if we now solve (ordered by number of bonds) successively (2.9) for the $f_\tau(p)$ of all $m + 2$ point graphs we find that the first power of p^{-1} that they will contain is p^{-m-1}, as we set out to show.

To summarize the result of this section

$$\ln W_p(G) = n \ln p + \sum_{\{\tau\}} T_{G,\tau} f_\tau(p) + O(p^{-m}),$$

where $W_p(G)$ is the number of distinct ways to color a graph G of n vertices with p colors, $\{\tau\}$ is all star subgraphs of G with (at most) m vertices. If $m \geqslant n$, the error term is, of course, zero.

III. APPLICATIONS INVOLVING THE INITIAL TERMS

In order to apply the results of the previous section we need a list of 1-, 2-,... point stars. Many orderly listings of graphs to various orders exist in the literature. For the convenience of the reader we reproduce a list of them through 5 points (Figure 1). The numbers attached to each graph in Figure 1 are taken from the list of Baker *et al* [5]. We will refer to the graphs by these numbers throughout. There is one graph with 2 points, one with 3, 3 with 4, 10 with 5 points, and 56 with 6 points. By contrast, there are for connected graphs 1, 2, 6, 21, and 112 graphs, respectively. When the separated configurations are added there are still more to be considered. Hence the labor of obtaining the required counts is greatly reduced, especially as the graphs eliminated commonly occur much more frequently than those retained.

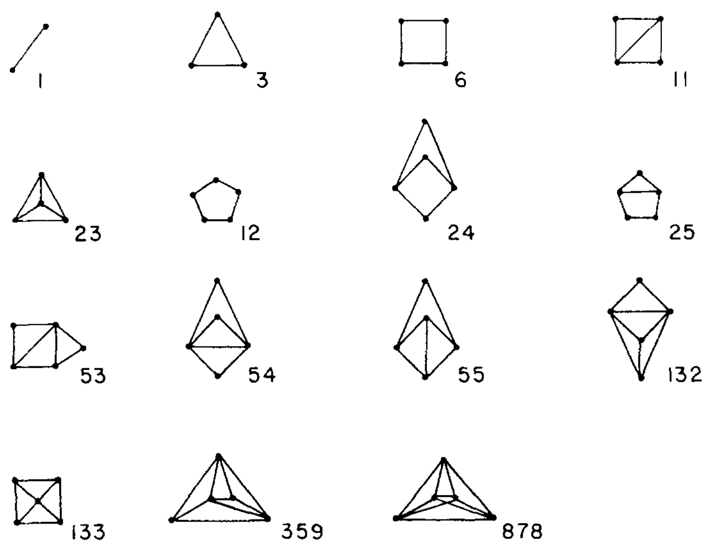

FIG. 1. All star graphs with 5 or fewer vertices.

The values of the $f_T(p)$ for the graphs given in Figure 1 are straightforward to calculate. They are

$$f_1(p) = \ln(1 - \sigma),$$

where we define, $\sigma \equiv p^{-1}$. For the polygons (numbers 3, 6, 12) we find

$$f_{pn}(p) = \ln[1 - (-\sigma/(1 - \sigma))^{n-1}], \tag{3.1}$$

where n is the number of vertices. For graphs of cyclomatic index 2

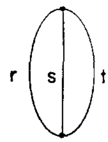

FIG. 2. Topologic structure of a θ-graph (cyclomatic index 2).

(θ-graphs in the notation of Sykes *et al.* [6]) we find (Figure 2)

$$f_\theta(p) = \ln\left\{\frac{\left(1 - \left(\frac{-\sigma}{1-\sigma}\right)^{r+s-1} - \left(\frac{-\sigma}{1-\sigma}\right)^{r+t-1} - \left(\frac{-\sigma}{1-\sigma}\right)^{s+t-1} + \frac{(1-2\sigma)(-\sigma)^{r+s+t-2}}{(1-\sigma)^{r+s+t-1}}\right)}{\left[1 - \left(\frac{-\sigma}{1-\sigma}\right)^{r+s-1}\right]\left[1 - \left(\frac{-\sigma}{1-\sigma}\right)^{r+t-1}\right]\left[1 - \left(\frac{-\sigma}{1-\sigma}\right)^{s+t-1}\right]}\right\} \tag{3.2}$$

where r, s, t are the number of edges in each of the three legs of the θ-graph and $r + s + t - 1$ is the number of vertices. Graphs number 11, 24, and 25 are of this type. It is to be noted that $f_{11}(p) = -f_6(p)$ and $f_{25}(p) = -f_{12}(p)$. This relation always holds if one leg is of length one, i.e., $f_\theta(p) = -f$ for the largest circumscribed polygon. One can work out general formulas successively for each of the four distinct topologies [7] of graphs of cyclomatic index 3, the 17 topologies [6] with cyclomatic index 4, etc. but we shall not do so as the number of representations of each type is small in the list of Figure 1. We shall rather give them explicitly:

$$f_{23}(p) = \ln\left[\frac{(1-3\sigma)(1-3\sigma+3\sigma^2)^3}{(1-\sigma)^6\,(1-2\sigma)^3}\right], \tag{3.3}$$

$$f_{53}(p) = +f_{12}(p), \qquad f_{54}(p) = -f_{24}(p), \tag{3.4}$$

$$f_{55}(p) = \ln\left[\frac{(1-\sigma)^8\,(1-3\sigma+3\sigma^2)(1-4\sigma+5\sigma^2)}{(1-2\sigma)(1-4\sigma+6\sigma^2-4\sigma^3)^2\,(1-5\sigma+10\sigma^2-7\sigma^3)}\right],$$
$$f_{132}(p) = -f_{55}(p). \tag{3.5}$$

We give the rest of the f's in the form of the defining equations which are computationally easier to use, and illustrate the method of derivation:

$$f_{133}(p) = \ln[(1-\sigma)(1-2\sigma)(1-5\sigma+7\sigma^2)] - 8f_1 - 4f_3 - 5f_6 - 4f_{11} - 4f_{12} - 2f_{24} - 4f_{25} - 4f_{53} - 4f_{55}\,, \tag{3.6}$$

$$f_{359}(p) = \ln[(1-\sigma)(1-2\sigma)(1-3\sigma)^2] - 9f_1 - 7f_3 - 9f_6 - 15f_{11} - 6f_{12} - 2f_{23} - 4f_{24} - 24f_{25} - 18f_{53} - 6f_{132} - 3f_{133}, \quad (3.7)$$

$$f_{878}(p) = \ln[(1-\sigma)(1-2\sigma)(1-3\sigma)(1-4\sigma)] - 10f_1 - 10f_3 - 15f_6 - 30f_{11} - 12f_{12} - 5f_{23} - 10f_{24} - 60f_{25} - 60f_{53} - 10f_{54} - 30f_{55} - 30f_{132} - 15f_{133} - 10f_{359}. \quad (3.8)$$

As an example of a problem for which estimates can easily be given by this method, but are scarcely possible from form (2.2) directly, consider the problem of coloring the vertices of a plane square lattice. The star graphs required are shown in Figure 3 up to eight points. The f's are given

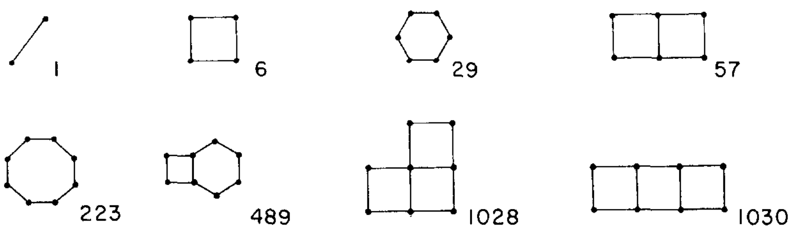

FIG. 3. All star graphs of 8 or fewer vertices which are embeddable on the square lattice.

by (3.1)–(3.2) except for the last two whose f's are equal to the f for number 223. Using the lattice constants [5] on a large square lattice of N vertices, we obtain

$$\frac{\ln W_{SQ}(p)}{N} = \ln p + 2f_1 + f_6 + 2f_{29} + 2f_{57} + 7f_{223} + 12f_{489} + 4f_{1028} + 2f_{1030} + \cdots \quad (3.9)$$

or substituting in for the f_i, we have

$$\begin{aligned} w_{SQ}(p) &= \lim_{N\to\infty} [W_{SQ}(p)]^{1/N} \\ &\approx p(1-\sigma)^2 \left(1 + \left(\frac{\sigma}{1-\sigma}\right)^3\right)\left(1 + \left(\frac{\sigma}{1-\sigma}\right)^7\right), \end{aligned} \quad (3.10)$$

where $\sigma = p^{-1}$. This expression yields, for $p = 3$, 1.51171875 as the eight-point approximation. The exact solution to this problem is available [8] and is

$$w_{SQ}(3) = (\tfrac{4}{3})^{3/2} \approx 1.5396007, \quad (3.11)$$

which differs from our estimate less than two per cent. The accuracy of our expression presumably increases as the number of colors increases, yielding progressively more accurate results. For example, we estimate

$$w_{SQ}(4) = 2.33440023, \tag{3.12}$$

where the eight-point terms have changed it from only the fourth figure on.

For some other lattices we can easily give estimates. For the honeycomb lattice (coordination number 3) we obtain, ignoring 11-point and higher graphs,

$$\frac{\ln W_H(p)}{N} = \ln p + \tfrac{3}{2}\ln(1-\sigma) + \tfrac{1}{2}\ln[1 + (\sigma/(1-\sigma))^5], \tag{3.13}$$

where the non-vanishing contributions come from graphs 1 and 29. The next order to contribute is the ten-point graphs, but they cancel each other. Numerically we find

$$w_H(3) = 1.6583124, \tag{3.14}$$

where corrections of the order of several thousandths are anticipated from higher order graphs. Likewise the diamond lattice (coordination number 4) coloring estimates can be readily given as

$$\frac{\ln W_D(p)}{N} = \ln p + 2\ln(1-\sigma) + 2\ln[1 + (\sigma/(1-\sigma))^5], \tag{3.15}$$

where numerically

$$w_D(3) = 1.4179687. \tag{3.16}$$

We have also carried the series for the simple cubic lattice through seven-point graphs. The additional stars required are shown in Figure 4.

FIG. 4. Seven vertex stars which are embeddable on the simple cubic lattice but not the square lattice.

Graph 158 is a θ-graph (Equation 3.2) and the result for 473 is

$$\begin{aligned} f_{473}(p) = {} & \ln[(1-\sigma)(1 - 8\sigma + 28\sigma^2 - 53\sigma^3 + 55\sigma^4 - 25\sigma^5)] \\ & - 9f_1(p) - 3f_6(p) - 4f_{29}(p) - 3f_{57}(p) - 3f_{158}(p). \end{aligned} \tag{3.17}$$

The asymptotic formula for the colorings of the simple cubic lattice is

$$\frac{\ln W_{SC}(p)}{N} = \ln p + 3f_1 + 3f_6 + 22f_{29} + 18f_{57} + 24f_{158} + 8f_{473} . \tag{3.18}$$

Numerically we estimate

$$w_{SC}(3) = 1.4969365; \tag{3.19}$$

however, we expect relatively substantial contributions from eight- and higher point graphs.

Finally we give the expression for the number of colorings of the triangular lattice including all six-point graphs. The additional graphs needed are shown in Figure 5. Their corresponding f functions are given simply by,

$$f_{58} = -f_{134} = -f_{135} = -f_{153} = f_{360} = f_{371} = f_{372} = -f_{29} . \tag{3.20}$$

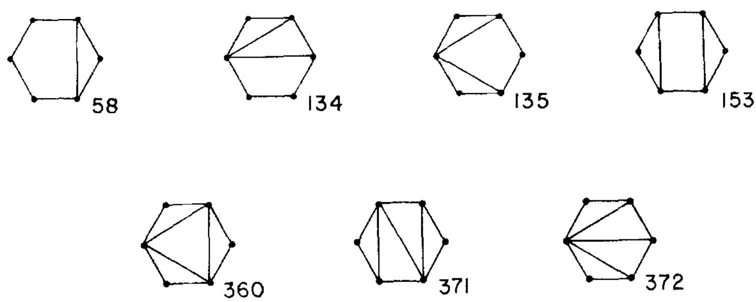

FIG. 5. Six vertex stars which are embeddable on the triangular lattice in addition to those in Figure 3.

Hence

$$\begin{aligned}\frac{\ln W_T(p)}{N} = \ln p &+ 3f_1 + 2f_3 + 3f_6 + 3f_{11} + 6f_{12} + 12f_{25} \\ &+ 6f_{53} + 15f_{29} + 12f_{57} + 30f_{58} + 24f_{134} + 12f_{135} \\ &+ 6f_{153} + 2f_{360} + 6f_{371} + 6f_{372} + \cdots .\end{aligned} \tag{3.21}$$

We have, substituting in the values,

$$\frac{\ln W_T(p)}{N} = \ln[p(1-\sigma)^3 [1 - (\sigma/(1-\sigma))^2]^2 [1 + (\sigma/(1-\sigma))^5]]. \tag{3.22}$$

Numerically, we obtain

$$w_T(4) = 1.3388203. \tag{3.23}$$

This is to be compared with the exact answer [9] of

$$\begin{aligned} w_T(4) &= \frac{2^2}{1.3}\frac{5^2}{4.6}\frac{8^2}{7.9}\cdots \\ &= 1.46099\cdots, \end{aligned} \tag{3.24}$$

which indicates an error of about nine per cent.

Special interest attaches to coloring estimates for those planar graphs in which every face is a triangle, every triangle a face, and every square subgraph (number 6) corresponds to exactly one square with diagonal (number 11). Rough estimates of the number of colorations for this class of planar graphs can be given with an error of p^{-4} (five-vertex graphs ignored). We find, using Euler's formula relating edges, faces, and vertices,

$$\ln W = N \ln p + (3N-6)\ln(1-1/p) + (2N-5)\ln\left(\frac{1-2/p}{(1-1/p)^2}\right) \tag{3.25}$$

or

$$W \sim \frac{p(p-1)^4}{(p-2)^5}\left[\frac{(p-2)^2}{(p-1)}\right]^N, \tag{3.26}$$

where N, the number of vertices, is three or more.

It will be noted that (3.26) is exact for $N = 3$, as expected. A special word is required about graph 23; no graph in the class considered has it as a subgraph, but it does occur as a graph. Equations (3.25)–(3.26) are only valid to order p^{-3} for this graph; however, when we remember that the correct number of counts must be an integer multiple of $4!\binom{p}{4}$ for $p \geqslant 4$, then we get the correct number of counts for all $p \geqslant 4$ from (3.26) for graph 23 by rounding to the nearest integer multiple (namely, one) of $4!\binom{p}{4}$.

IV. Reduction of the Series to the Strong Embedding System

A further simplification of the expansion for $\ln W_p(G)$ can be obtained by reexpressing our results, not in terms of the weak embeddings used in Section II, but in terms of the strong embedding set of graph constants. To this end we define the matrix $S_{G,\tau}$ to be the number of distinct ways an unlabeled subgraph τ can be embedded on G such that every edge of τ corresponds to an edge of G, and, if any two vertices of τ lie on vertices of G which are joined in G by an edge, then they are also joined in τ by an edge.

We now quote the following Theorem of Sykes *et al.* [6] relating S and T.

THEOREM 4. (Sykes, Essam, Heap and Hiley).

$$T_{G,\tau} = \sum_{\tau'} T_{\tau',\tau} S_{G,\tau'} . \tag{4.1}$$

Here the summation extends over all graphs τ' with the same number of vertices as τ.

We remark that τ' is necessarily connected if τ is, since τ would then be a connected subgraph of the same number of vertices. Substituting equation (4.1) into (2.9) we obtain

THEOREM 5.

$$\ln Z_p(G) = \sum_{\tau} S_{G,\tau} F_\tau(p). \tag{4.2}$$

Here the summation is only over star subgraphs. Further, for large positive real p

$$F_\tau(p) = O(p^{-(n_\tau - 1)}), \tag{4.3}$$

where n_τ is the number of vertices in τ.

To see these we need only note that

$$F_{\tau'}(p) = \sum_{\tau} T_{\tau',\tau} f_\tau(p). \tag{4.4}$$

As the summation is only over τ with the same number of vertices as τ', relation (2.10) is not destroyed, and hence (4.3) follows.

The simplification implicit in Theorem 5 is twofold. First, the number of strong embeddings is much smaller than the weak ones. Second, the F's vanish for a wider class of graphs than do the f's. This class is described by the following theorem, which is modeled after Theorem VI of Essam and Sykes [10].

THEOREM 6. *Suppose a graph G can be divided into two disconnected components by deleting a set of n vertices (cut-set) and further that the section graph of these n vertices is the complete n-graph, then* $F_G(p) = 0$.

To establish this theorem we proceed by double induction—first on n and then on the size of graph considered. By Theorem 2, this theorem is true for $n = 1$ and graphs of any size. Now assume it is true for $n \leqslant N$. If then not every point of a cut-set of size $N + 1$ touches some point of both separated components G_1 and G_2, then there must be at least one point which touches only one component, say G_1. By affixing it to G_1 we may define a new cut-set of N (or fewer) points which separates G. The theorem will be true for such G. Hence we need only to consider those cases in which all $N + 1$ points of the cut-set C are connected to some point in both components. (The smallest such graph contains one point in G_1 and in G_2.) The number of colorations is easily evaluated as

$$W_p(G) = \frac{W_p(G_1 \cup C)\, W_p(G_2 \cup C)}{W_p(C)}, \tag{4.5}$$

as no point in G_1 touches any point in G_2. Now, any strongly embedded subgraph which has a point in G_1 and G_2 must have, as its intersection with C, a complete graph of $m \leqslant N + 1$. If $m \leqslant N$, then its F is zero by induction. If $m = N + 1$, it is smaller than G as it is a subgraph. The smallest such G, $\hat{G}$, has only one point in G_1 and one in G_2. As it has no strongly embeddable subgraph with at least one point in each of G_1 and G_2, all its strongly embedded subgraphs with non-zero F are in $G_1 \cup C$ or $G_2 \cup C$. It therefore follows that

$$F_\tau(p)\, S_{\hat{G},\tau} = F_\tau(p)[S_{G_1 \cup C,\tau} + S_{G_2 \cup C,\tau} - S_{C,\tau}]. \tag{4.6}$$

By (4.5) and (2.3)

$$\ln Z_p(\hat{G}) = \ln Z_p(G_1 \cup C) + \ln Z_p(G_2 \cup C) - \ln Z_p(C). \tag{4.7}$$

Applying (4.2) and (4.6) to the left side of (4.7) and (4.2) to the right side, we obtain

$$F_{\hat{G}}(p) = 0. \tag{4.8}$$

Now assume that (4.6) is true for all graphs smaller than the one we are considering. Again, any strongly embedded subgraph which has a point in G_1 and in G_2 must intersect C with a complete graph. Hence by induction the F for this graph vanishes. Hence (4.6) must hold for this G. Therefore we may again conclude (4.8) for any graph G with a cut set of $N + 1$ points. Hence by induction Theorem 6 is true.

We tabulate here the F's for the graphs illustrated in the previous section. The F's for the line, and polygons are unchanged. Thus,

$$\begin{aligned} F_1(p) &= \ln(1 - \sigma), \\ F_{pn}(p) &= \ln[1 - (-\sigma/(1 - \sigma))^{n-1}]. \end{aligned} \tag{4.9}$$

Those for the θ-graphs are modified as

$$\begin{aligned} F_\theta(p) &= 0, && r, s, \text{ or } t = 1, \\ F_\theta(p) &= f_\theta(p) && \text{otherwise,} \end{aligned} \tag{4.10}$$

where $f_\theta(p)$ is given in (3.2). Of the others, numbers 11, 25, 53, 54, 57, 58 132, 134, 135, 153, 359, 360, 371, 372, 489, 1028, and 1030 all vanish on account of Theorem 6, or 17 of the 30 considered in the weak system. In addition, there are no strong embeddings of number 158 on the simple cubic lattice. Thus the remaining relevant strong system F's are

$$\begin{aligned}
F_{23} &= \ln\left[\frac{(1-\sigma)^3\,(1-3\sigma)}{(1-2\sigma)^3}\right],\\
F_{55} &= \ln\left[\frac{(1-\sigma)^4\,(1-4\sigma+5\sigma^2)}{(1-2\sigma)(1-3\sigma+3\sigma^2)^2}\right],\\
F_{133} &= \ln\left[\frac{(1-\sigma)^4\,(1-5\sigma+7\sigma^2)}{(1-2\sigma)^3\,(1-3\sigma+3\sigma^2)}\right],\\
F_{473} &= \mathrm{lu}\left[\frac{(1-\sigma)^6\,(1-8\sigma+28\sigma^2-53\sigma^3+55\sigma^4-25\sigma^5)}{(1-3\sigma+3\sigma^2)^3\,(1-5\sigma+10\sigma^2-10\sigma^3+5\sigma^4)}\right],\\
F_{878} &= \ln\left[\frac{(1-2\sigma)^6\,(1-4\sigma)}{(1-\sigma)^4\,(1-3\sigma)^4}\right].
\end{aligned} \tag{4.11}$$

The estimates corresponding to those of Section III are simplified to

$$\begin{aligned}
\frac{\ln W_{SQ}(p)}{N} &= \ln p + 2F_1 + F_6 + F_{223}\,,\\
\frac{\ln W_H(p)}{N} &= \ln p + \tfrac{3}{2}F_1 + \tfrac{1}{2}F_{29}\,,\\
\frac{\ln W_D(p)}{N} &= \ln p + 2F_1 + 2F_{29}\,,\\
\frac{\ln W_{SC}(p)}{N} &= \ln p + 3F_1 + 3F_6 + 4F_{29} + 8F_{473}\,,\\
\frac{\ln W_T(p)}{N} &= \ln p + 3F_1 + 2F_3 + F_{29}\,.
\end{aligned} \tag{4.12}$$

We note that the class of graphs which contributes to the coloring problem is the same as that which is involved in the site percolation problem as discussed by Essam and Sykes [10]. They listed all the graphs required to extend the strong embedding series through 10 points on the triangular lattice. There are one 7-point star, one 9-point, two 10-point stars and the eight-, nine-, and ten-sided polygons to that order. Through seven points we have

$$\frac{\ln W_T(p)}{N} \approx \ln[p(1-2\sigma)^{-3}\,(1-9\sigma+31\sigma^2-49\sigma^3+31\sigma^4)], \tag{4.13}$$

which yields numerically

$$w_T(4) \approx 1.375. \tag{4.14}$$

We have also added the eight- and nine-point terms which yield, numerically,

$$w_T(4) \approx 1.389657, \tag{4.15}$$

which differs from the result of Baxter [9] by about five per cent.

We remark, finally, that the derivation of the F's themselves can be greatly simplified by the use of Theorem 5, and the two facts that (i) p divides every chromatic polynomial, and (ii) $(p - 1)$ divides every chromatic polynomial for a star graph. Since a chromatic polynomial for an n-point graph is of degree n, and (4.2) of Theorem 5, by (4.3) determines completely the coefficients of $p^n, p^{n-1}, \ldots, p^2$ in terms of the F's for subgraphs τ, we may use (i) and (ii) to determine the coefficient of p and show that the constant vanishes. Plainly, from the matrix $S_{G,\tau}$ alone, we can therefore construct all the F's and hence also all the Z's, and finally the number of colorations function $W_p(G)$.

Note added in proof. Since writing this article, several additional relevant references have come to the attention of the author. Whitney [11] has anticipated Theorems 2 and 3 by a lengthier method and had previously made the application (3.25). Tutte [12] has proved theorems analogous to theorems 3 and 5 for the rank function. Nagle [13] has recently developed a graphical expansion in powers of $(p - 1)^{-1}$ instead of p^{-1}.

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